## **10. Heat transfer under the heat source**

The general aim of this lecture is the analysis of the heat transfer phenomenon under the heat source. However, at first, we consider the without the heat source, but with non-homogeneous boundary conditions. Particularly, we suppose that the heat flux at the ends of the body is given. Therefore, we have the homogeneous heat equation with non-homogeneous second order boundary conditions. Changing the unknown function, we transform this problem to the non-homogeneous equation with homogeneous boundary conditions. The analogical system is the mathematical model of the heat transfer phenomenon under the heat source. This problem is solved by Fourier method. The solution of this system is found as a Fourier series. We find the corresponding Fourier coefficients using the given initial conditions. The heat transfer phenomenon under the heat source is considered as example.

### **10.1. Heat transfer with known heat flux at the ends**

Consider again the heat transfer phenomenon for the long thin body. We have the heat equation

 *ut = a2 uxx*, 0 < *x* < *L*, *t* > 0. (10.1)

The initial temperature *ϕ* =*ϕ*(*x*) of the body is given, i.e. we have the initial conditions

 *u*(*x*,0) = *ϕ*(*x*), 0 < *x* < *L*. (10.2)

Suppose the heat fluxes at the left end *p = p*(*t*) and at the right end *q = q*(*t*) are given, i.e. we have non-homogeneous second order boundary conditions

 *ux*(0,*t*) = *p*(*t*), *ux*(*L*,*t*) = *q*(*t*), *t* > 0. (10.3)

The system (10.1) – (10.3) is the mathematical model of the considered phenomenon.

We cannot to use the method of separation of variables because of the boundary conditions. Try to find the solution of this problem as the sum

 *u*(*x*,*t*) = *v*(*x*,*t*) + *w*(*x*,*t*), (10.4)

where we choose the function *w* such that it satisfies the boundary conditions (10.2). Suppose the spatial derivative of the function *w* is linear, i.e.

*wx*(*x*,*t*) = *α*(*t*)+*β*(*t*)*x*.

We choose the functions *α* and *β* for obtaining the equalities (10.2). We have

*wx*(*x*,0) = *α*(*t*) = *p*(*t*),

*wx*(*L*,*t*) = *α*(*t*)+*β*(*t*)*L = q*(*t*).

Then we determine

*α*(*t*) = *p*(*t*), *β*(*t*) *=* [*q*(*t*) – *p*(*t*)]*x*/*L*.

Now we obtain

*wx*(*x*,*t*) *= p*(*t*) *+* [*q*(*t*) – *p*(*t*)]*x*/*L*.

Integrate this equality. We get

*w*(*x*,*t*) *= p*(*t*)*x +* [*q*(*t*) – *p*(*t*)]*x*2/2*L* + *c*,

where the constant *c* is arbitrary. For all values of this constant, the function *w* satisfies the boundary conditions (10.2). Therefore, the function *v* of the equality (10.4) satisfies the homogeneous second order boundary conditions. We consider easiest case, where *c =* 0. Then we choose the function

 *w*(*x*,*t*) *= p*(*t*)*x +* [*q*(*t*) – *p*(*t*)]*x*2/2*L.* (10.5)

Using the formula (10.5), put the function *u* from the equality (10.4) to the equation (10.1). We have

*vt* + *p*'(*t*)*x +* [*q*'(*t*) – *p*'(*t*)]*x*2/2*L*] *= a*2*vxx* + [*q*(*t*) – *p*(*t*)]/*L*.

Therefore, we have the equality

 *vt = a*2*vxx* + *f*(*x*,*t*), (10.6)

where

*f*(*x*,*t*) = [*q*(*t*) – *p*(*t*)]/*L* – {*p*'(*t*)*x +* [*q*'(*t*) – *p*'(*t*)]*x*2/2*L*]}

Now we put the function *u* from the equality (10.4) to the equation (10.3). We get

*v*(*x*,0) + *p*(0)*x +* [*q*(0) – *p*(0)] *x*2/2*L* = *ϕ*(*x*),

Then we obtain the initial conditions for the function *v*

 *v*(*x*,0) = *ϕ*1(*x*), (10.7)

where

*ϕ*1(*x*) = *ϕ*(*x*) – {*p*(0)*x +* [*q*(0) – *p*(0)] *x*2/2*L* *p*(0)}*.*

By choosing of the function *w* the boundary conditions for the function *v* is homogeneous

 *vx*(*x*,0) =0, *v x*(*x*,0) = 0. (10.8)

Thus, the given problem (10.1) – (10.3) is transformed to the non-homogeneous heat equation (10.6) with initial conditions (10.7) and homogeneous boundary conditions (10.8). If we find the solution of this problem, then we will find the solution of the initial problem (10.1) – (10.3) by the formula (10.4). Note that the problem (10.6) – (10.8) has a direct physical sense.

### **10.2. Mathematical model of the heat transfer under the heat source**

Consider the heat transfer phenomenon for the thin long body under the heat source. This phenomenon is by the non-homogeneous heat equation

 *ut = a*2*uxx* + *f*(*x*,*t*), 0 < *x* < *L*, *t* > 0, (10.9)

where the function *f* characterizes the influence of the heat source. Suppose the ends of the body are isolated. Then we have the homogeneous second order boundary conditions

 *ux*(0,*t*) = 0, *u x*(*L*,*t*) = 0, *t* > 0. (10.10)

The initial temperature *ϕ* =*ϕ*(*x*) is given. Then we have the initial conditions

 *u*(*x*,0) = *ϕ*(*x*), 0 < *x* < *L*. (10.11)

We have the second homogeneous boundary problem for the non-homogeneous heat equation.

### **10.3. Fourier method**

We know that the corresponding boundary problem for the homogeneous boundary problem has the solution that is represented as a cosine Fourier series. We try to find the solution of the problem (10.9) – (10.11) in the analogical form, i.e.

  (10.12)

where the functions *uk*that are Fourier coefficients are unknown. Note that for all *uk* the function *u* of the formula (10.12) satisfies the boundary conditions (10.10). We try to find the functions *uk*such that the function *u* determined by the formula (10.12) satisfies the equation (10.9) and the boundary conditions (10.11).

Put the function *u* from the equality (10.12) to the equation (10.9). We get



Multiply this equality by the function  and integrate the result by *x* from 0 to *L.* We have

 (10.13)

Find the integral



Determine



If  we can find



For *k = n* and  we get



Finally, for *k = n =* 0 we obtain



Putting the results to the formula (10.13), we obtain

  (10.14)

where

  (10.15)

If the function *un* satisfies the equation (10.14), then the function *u* determined by the formula (7.12) is the solution of the non-homogeneous heat equation (10.9) with boundary conditions (10.10).

Put the function *u* from the formula (10.12) to the initial conditions (10.11). We get



Multiply this equality by the function  and integrate the result by *x* from 0 to *L.* We have



Using the previous results, determine

   (10.16)

where

  (10.17)

If the function *un* satisfies the conditions (10.16), then the function *u* determined by the formula (10.12) satisfies the initial conditions (10.11).

Thus, it is necessary to solve the problem (10.14), (10.16) for determining the Fourier coefficient *un*.

### **10.4. Finding the solution of the problem**

Find the solution of the problem (10.14), (10.16). Determine the value



because of the equality (10.14). After integration we have



Using the equality (10.16), we find



Put this value to the formula (10.12). We determine the solution of the problem (10.9) – (10.11)

  (10.18)

Thus, the solution of the considered problem is determined by the formula (10.18), where the Fourier coefficients of the given functions are determined by the formulas (10.15) and (10.17).

We can transform this result. Put the values of the Fourier coefficients of the given functions are determined by the equalities (10.15) and (10.17) to the formula (10.18). We get



Determine the ***Green function***

  (10.19)

Then the solution of the problem (10.9) – (10.11) is determined by the formula

  (10.20)

We obtain the direct dependence of the solution of the given boundary problem from its initial state the function of the heat source. Note that this Green function was be obtained before for the second order boundary problem for the heat equation without heat source.

### **10.5. Example**

Consider the partial case of the problem (10.9) – (10.11). Let us analyze the heat transfer in the body of the length *L=π* with coefficient *a =* 1. Suppose the source determines by the function
*f*(*x*,*t*) = cos *x*. Then we have the non-homogeneous heat equation

 *ut = uxx* + sin *x*, 0 < *x* < *π*, *t* > 0. (10.21)

The ends of the body are isolated. Then we have the boundary conditions

 *ux*(0,*t*) = 0, *ux*(*π*,*t*) = 0, *t* > 0. (10.22)

Suppose the initial temperature of the body is zero. Then we have the initial conditions

 *u*(*x*,0) = 0, 0 < *x* < *L*. (10.23)

Using the formula (10.18), determine the solution of the problem (10.21) – (10.23) by the formula



Using the formulas (10.15) and (10.17), find the Fourier coefficients of the given functions



Thus, the solution of the problem (10.21) – (10.23) is

  (10.24)

Check that this is, in reality, the solution of the considered problem. Find the value
*u*(*x*,0) = 0, i.e. the initial condition is true. Using the equality



determine the truth of the boundary conditions (10.22). Finally, find



Then the function *u* determined by the formula (10.24) is the solution of the considered problem.

Give now the physical interpretation of the obtained result. At first, return to the problem statement. We have the thermally isolated body. The distribution of the temperature is uniformly at the initial time. There exists the constant heat source that is positive at the left part of the body and negative at its right side. Consider the result of analysis (see Figure 10.1).



Figure 9.1. Temperature distribution.

We have the uniformly temperature distribution at the initial time. Then the temperature increase at the left side of the body, and decrease at its right side, besides the velocity of change is maximal at the ends of the body and zero at the its middle. The velocity of change of the temperature decrease by time and tends to zero, if the time tends to infinity. The temperature distribution tends to the state of equilibrium that is the function cos *x.*

 Explain the obtained result. We consider the body under constant heat source that is positive at the left side of the body and negative at its right side. By the influence of this heat source, the temperature at the left side of the body increases, and the temperature at its right side decreases. The body is isolated. Therefore, we have no heat exchange with environment. However, the temperature at the ends of the body become different. Then we have the heat flux from the hot left side to the cold right side of the body. This flux increase by time, because the temperature difference at the ends increases.

Now we have two factors of the temperature change. There are the constant influence of the heat source end the increased influence of the heat flux. This influence is opposite. Particularly, the temperature at the left increases by the positive heat source and decreases, because the heat moves from the left to the right. At the initial time, we do not have the negative factor, because of the equality of the temperature at the ends of the body. The positive factor is constant, because the source does not depends from time. However, the negative factor increase by increasing of the temperature difference. Therefore, the velocity of temperature increasing decreases and tends to zero by time.

We have the inverse situation at the right side of the body. The temperature changes there by the negative influence of the heat source and the positive influence of the heat flux. At the initial time, we do not have the positive factor, because of the equality of the temperature at the ends of the body. The negative factor is constant, because the source does not depends from time. However, the positive factor increase by increasing of the temperature difference. Therefore, the velocity of temperature increasing decreases and tends to zero by time. We have the state of equilibrium, if the time tends to the infinity.

###  **Conclusions**

* The heat equation with non-homogeneous boundary conditions can be transformed to the non-homogeneous heat equation with homogeneous boundary conditions.
* The heat transfer phenomenon under the heat source is described by the non-homogeneous heat equation.
* The solution of the problem is represented as a Fourier series by the Fourier method.
* The Fourier coefficients of solution depend from the time and satisfy non-homogeneous first order ordinary differential equations with initial conditions.
* The parameters of the obtained system are Fourier coefficients of the source function and initial state.
* The direct dependence of the problem solution from the heat source and the initial state can be determined using the Green function that depends from the concrete body and does not depends from the heat source and the initial temperature.
* The concrete heat transfer phenomenon is considered as example.

### Task. **Heat transfer under the heat source.**

Consider the heat transfer under the exterior heat source characterized by the given function *f.* This phenomenon is described by non-homogeneous heat equation

*ut = a2 uxx* + *f*(*x*,*t*), 0 < *x* < *L*, *t* > 0.

Suppose the initial temperature is zero. Then we have the initial condition

*u*(*x*,0) = 0, 0 < *x* < *L*.

The temperature or the heat flux are zero at the ends, i.e. we have one of the following boundary conditions

 *u*(0,*t*) = 0, *u*(*π*,*t*) = 0, *t* > 0; (\*)

 *ux*(0,*t*) = 0, *ux*(*π*,*t*) = 0, *t* > 0. (\*\*)

Table of parameters

|  |  |  |  |  |
| --- | --- | --- | --- | --- |
| variant | boundary condition | *L* | *a* | *f* |
| 1 | \*\* | π | 1 | cos *x* |
| 2 | \* | 1 | 2 |  –sin π*x* |
| 3 | \* | 1 | 3 | sin 2π*x* |
| 4 | \* | π | ½ | –2 sin *x* |
| 5 | \*\* | 1 | 1 | –cos π*x* |
| 6 | \*\* | π | ½ | cos 2*x* |
| 7 | \* | π | ½ | sin 2*x* |
| 8 | \*\* | 1 | 2 | cos 2π*x* |

Task:

1. Determine the solution of the problem as sinus Fourier series for the boundary conditions (\*) and cosine Fourier series for the boundary conditions (\*\*).
2. Find the Fourier coefficient of the parameters of the system.
3. Solves ordinary differential equations with respect to the Fourier coefficients of the solution of the problem.
4. Check that this is, in reality the solution of the boundary problem.
5. Show the graph (temperature distribution for the different time points).
6. Give the physical interpretation of the results.